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Gravitational radiation from a particle in bound orbit around a black hole; relativistic correction

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Abstract. Gravitational radiation from a system of two body, one as test particle and other as black hole (we assume, m_1 is mass of the test particle and m_2 is mass of black hole in bound orbits (orbital eccentricities $e < 1$) and $E^2 < 1$; E is the energy, is calculated with relativistic correction using the method of inertia tensor and multipole formalism. Plots of power versus eccentricity of the bound orbit of first kind are presented, and average total power radiated as a function of eccentricity is plotted according to inertia tensor method. According to multipole formalism the power radiated in gravitational waves from an bound orbit is given by enhancement factor $g(n,e)$ times the function of other parameters is plotted. The calculations apply for arbitrary eccentricity of the relative orbit, assuming orbital velocities are small.

1. Introduction

Accelerating heavenly bodies produces a disturbance in space-time which travels with the speed of light, called as gravitational waves. It is nothing but the perturbations of the flat space-time [5]. One of the most promising astrophysical source of gravitational radiation is the binary star system [4]. This system consists of binary NS, binary BH, and NS-BH binary. The binary star inspiraling each other radiates GR and come closer to each other. Then after certain time coalesce. The time of coalescence depends directly on the eccentricity of the orbit. The power of emitted radiation depends on the mass of black hole and the mass of the particle. It also depends on the semi-major axis and the eccentricity of the orbit before coalescence. Here in this work we used two method to calculate the power generated by point particle in bound orbit around the black hole.

2. Methodology

I. *Inertia Tensor Method:* Let the masses m_1 and m_2 have coordinates $(d_1 \cos \chi, d_1 \sin \chi)$ and $(-d_2 \cos \chi, -d_2 \sin \chi)$ in the xy plane as shown in figure 1. The origin will be taken to be at the center of mass so that,

$$d_1 = \left(\frac{m_2}{m_1 + m_2} \right) d, \quad d_2 = \left(\frac{m_1}{m_1 + m_2} \right) d \quad (1)$$



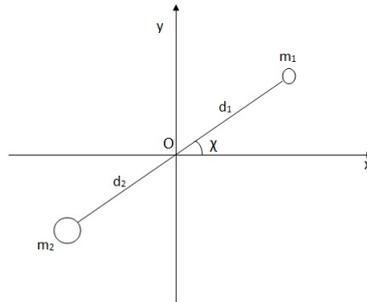


Figure 1. Coordinate system used in calculation.

Now the non-vanishing quadrupole tensor [3] are,

$$Q_{xx} = \mu d^2 \cos^2 \chi, \quad Q_{yy} = \mu d^2 \sin^2 \chi, \quad Q_{xy} = Q_{yx} = \mu d^2 \sin \chi \cos \chi \quad (2)$$

where, μ is the reduced mass of the binary system and the distance between two masses $d = d_1 + d_2$. From kepler motion, the orbit equation is,

$$u = \frac{1}{l}(1 + e \cos \chi) \quad (3)$$

where, $u = 1/d$, e is the eccentricity of the orbit and l is the latus rectum of the orbit.

We have equation with relativistic correction for bound orbit [1],

$$\pm \frac{d\chi}{d\phi} = (1 - 6\mu + 2\mu e)^{1/2} (1 - k^2 \cos^2 \chi/2)^{1/2} \quad (4)$$

where, k^2 depends on reduced mass of the system and eccentricity of the orbit and given by,

$$k^2 = \frac{4\mu e}{1 - 6\mu + 2\mu e}$$

Now, we can calculate the angular velocity ($\dot{\chi}$), with substitution of k^2 , $\cos^2 \chi/2$ and $d\phi/dt$ is,

$$\frac{dt}{d\phi} = \frac{E}{\mathbf{L}u^2(1 - 2Mu)} \quad (5)$$

where, \mathbf{L} denotes the angular momentum about an axis normal to the invariant plane and $M = m_1 + m_2$ i.e total mass of the binary system. Simplifying, we get angular velocity,

$$\dot{\chi} = \sqrt{1 - 6\mu - 2\mu e \cos \chi} \frac{\mathbf{L}(1 + e \cos \chi)^2}{El^2} \left[1 - \frac{2M}{l} - \frac{2Me \cos \chi}{l} \right] \quad (6)$$

We know that, $l = a(1 - e^2)$ where, a is the semimajor axis and e is the eccentricity of the ellipse(orbit). The total power is calculated from [3],

$$P = \frac{G}{5c^5} \left[\left(\frac{d^3 Q_{xx}}{dt^3} \frac{d^3 Q_{xx}}{dt^3} + 2 \frac{d^3 Q_{xy}}{dt^3} \frac{d^3 Q_{xy}}{dt^3} + \frac{d^3 Q_{yy}}{dt^3} \frac{d^3 Q_{yy}}{dt^3} \right) - \frac{1}{3} \left(\frac{d^3 Q_{xx}}{dt^3} \frac{d^3 Q_{xx}}{dt^3} + 2 \frac{d^3 Q_{xx}}{dt^3} \frac{d^3 Q_{yy}}{dt^3} + \frac{d^3 Q_{yy}}{dt^3} \frac{d^3 Q_{yy}}{dt^3} \right) \right] \quad (7)$$

Where, G is gravitational constant = $6.67 \times 10^{-11} N.m^2/kg^2$ and Q_{ij} are quadrupole tensors.

II. *Multipole Expansion*: In this method, we evaluate the m_{2M} , in terms of Q_{ij} they are [2],

$$m_{2\pm 2} = \frac{ik\omega^3}{10\sqrt{3}} \left(\frac{15}{32\pi} \right)^{1/2} (Q_{xx} - Q_{yy} \pm 2iQ_{xy}) \quad (8)$$

$$m_{2\pm 1} = 0 \quad (9)$$

$$m_{20} = \frac{-ik\omega^3}{10\sqrt{3}} \left(\frac{5}{16\pi} \right)^{1/2} (Q_{xx} + Q_{yy}) \quad (10)$$

Where, $k = \sqrt{32\pi G}$ and $\omega = n\omega_0$, $\omega_0 = [G(m_1 + m_2)/a^3]^{1/2}$ is the average angular velocity. Q_{ij} are calculated by using Fourier series of argument M with recursion relation of Bessel's function. Finally, the power radiated in the n^{th} harmonics is,

$$P(n) = \frac{32G^4 m_1^2 m_2^2 (m_1 + m_2)}{5c^5 a^5} g(n, e) \quad (11)$$

Where,

$$g(n, e) = \frac{n^4}{32} ([J_{n-2}(ne) - 2eJ_{n-1}(ne) + \frac{2}{n}J_n(ne) + 2eJ_{n+1}(ne) - J_{n+2}(ne)]^2 + (1 - e^2)[J_{n-2}(ne) - 2J_n(ne) + J_{n+2}(ne)]^2 + \frac{4}{3n^2}[J_n(ne)]^2) \quad (12)$$

3. Results

1. Inertia Tensor Method: It is straightforward to calculate the d^3Q_{ij}/dt^3 : and we have,

$$\frac{d^3Q_{xx}}{dt^3} = \frac{d^3Q_{xx}}{d\chi^3} \left(\frac{d\chi}{dt} \right)^3 + \frac{d^2Q_{xx}}{d\chi^2} \frac{d}{dt} \left(\frac{d\chi}{dt} \right)^2 + \frac{d^2Q_{xx}}{d\chi^2} \frac{d\chi}{dt} \frac{d^2\chi}{dt^2} + \frac{dQ_{xx}}{d\chi} \frac{d^3\chi}{dt^3}$$

and similar results for other components.

We can calculate angular velocity ($\dot{\chi}$), using equation (6). Since, $l = a(1 - e^2)$, taking semimajor axis (a) of an ellipse as constant, we vary eccentricity and after solving equation [6],

$$\frac{1}{L^2} = \frac{1}{lM} [1 - \mu(3 + e^2)]$$

We get,

$$L = l \sqrt{\frac{m_1 m_2}{(m_1 + m_2) - 3m_1 m_2 - m_1 m_2 e^2}}$$

We can calculate the latus rectum for ellipse, keeping semimajor axis as a constant. We can take $a = 11.458$, $m_2 =$ mass of black hole = 1, $m_1 =$ mass of test particle = 0.001

For bound orbit, $e < 1$ and $E^2 < 1$, to meet this requirement we put $E = 8/9$. The time period of the orbit, according to keplers law is,

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

Substituting $G = 1$ and $a = 11.458$ for simplicity, we get $1/T = 0.0041$.

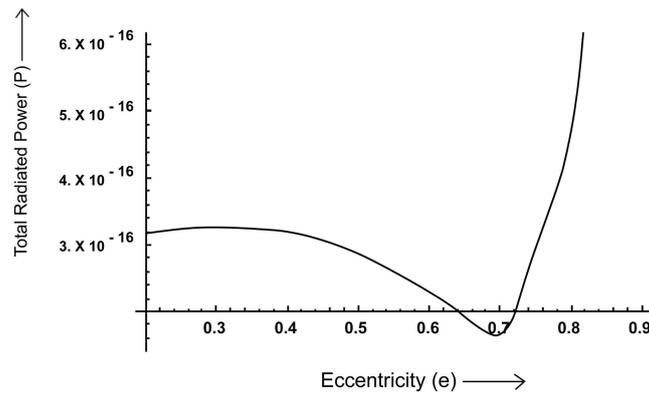


Figure 2. The relative power radiated with different eccentricities (e)

We calculated the numerical values of latus rectum and angular momentum(\mathbf{L}) for different eccentricities. We put the numerical values for mass(M), eccentricity(e), latus rectum (l), angular momentum then we obtained total average radiated power. We take only real part of our calculation, because imaginary part is very-very small; at the order 10^{-48} . The total relative radiated power is plotted with eccentricity(e), which is shown in fig 2.

2. Multipole Expansion: From this method, we have calculated the total power radiated and we plot it with different $[g(n, e)]$: that is n^{th} harmonics, for eccentricities $e = 0.2, 0.5, 0.7$. We get the following plot; fig(3) using this method.

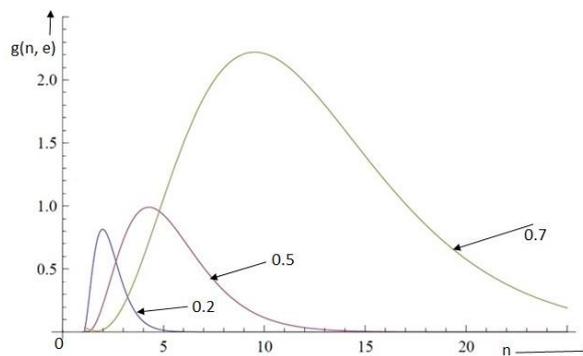


Figure 3. $[g(n, e)]$, the relative power radiated into the n^{th} harmonics, for eccentricities $e = 0.2, 0.5, 0.7$

4. Conclusion

1. Inertia Tensor Method

We calculated the relative total power radiated with the application of relativistic correction in classical results. We found that the total radiated power is strongly depend on eccentricity of the orbit. With relativistic correction, we found that with increasing of eccentricity, total radiated power is also increasing, but on further increasing of eccentricity we found certain decrement in total power radiated, looks local minimum in figure 2. Again, on further increasing in eccentricity, the total relative power is steeply rising. This is from the relativistic correction

in classical results. The nature of the plot is similar to the results of P.C. Peter's and J. Mathew's result's [3] except that the minimum doesn't appear in Newtonian result. We believe this minimum is due to relativistic correction. The radiation should depend so strongly on the eccentricity is not surprising. In electromagnetic radiation, the power radiated increases for increasing accelerations. Thus, the bodies will radiate most at their closest approach, and for fixed energy the higher the eccentricity, the higher the power radiated will be.

2. Multipole Expansion

From this method we calculate total radiated power in terms of the $g(n, e)$; the relative power radiated into the n^{th} harmonic. The plot of $g(n, e)$ with different eccentricities $e = 0.2, 0.5, 0.7$ are shown in figure (3), with the increasing of harmonics (n) and eccentricity (e), we get large and smooth curve. This explains why the higher harmonics dominate the radiation for (e) near 1. Fourier components of large (n) must be present to give such a peaking of the radiation at one part of the path.

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References

- [1] Chandrasekhar S., *The Mathematical Theory of Black Holes*, (Oxford University press, Walton Street, Oxford, 1983), chapter 3.
- [2] Mathews J., *J. Soc. Ind. Appl. Math.* **10**, 768 (1962).
- [3] Peters P. C. and Mathews J., 1963, *Phys. Rev.*, **131**, 435-440.
- [4] Schutz B. F., *Gravitational Radiation*, (arXiv:gr-qc/0003069 v1 16 Mar 2000).
- [5] Schutz B. F., *A First Course in General Relativity*, (Cambridge University press, 1985), chapter 9.
- [6] Weinberg S., *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley and Sons Inc., U.K., 2008), chapter 10.